

# Quantization of conical spaces in 3D gravity

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**ABSTRACT:** We discuss the quantization and holographic aspects of a class of conical spaces in 2+1 dimensional pure AdS gravity. These appear as topological solitons in the Chern-Simons formulation of the theory and are closely related to the recently studied conical solutions in higher spin gravity. We discuss the classical fluctuations around these solutions, which form exceptional coadjoint orbits of the asymptotic Virasoro group. We argue that the quantization of these solutions leads to nonunitary representations of the Virasoro algebra, on account of their having boundary graviton fluctuations which lower the energy. We propose a framework to quantize them in a semiclassical expansion in the inverse central charge, which we use to compute their one-loop corrected energies. Interestingly, the resulting Virasoro representations contain a null vector, thus providing an appearance of Kac's degenerate representations, which are nonunitary at large central charge, in the context of gravity. We match the computed quantum corrections in the bulk with the properties of a class of primaries in Kac's classification.

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## 1. Introduction

Pure gravity in 2+1 dimensions with negative cosmological constant has proven to be an interesting laboratory to test ideas in quantum gravity and holography. Despite having no local degrees of freedom, it possesses an asymptotic Virasoro symmetry [1] which can be viewed as the symmetry of a dual 1+1 dimensional CFT [2]. Furthermore the gravity theory contains black hole solutions [3] whose entropy can be understood from modular invariance of the dual CFT [4]. Despite these qualitative successes, it is still unclear whether pure gravity at weak coupling (large central charge  $c$ ) exists as a quantum theory and, if so, what are the properties of the dual CFT [5],[6],[7].

An important step in identifying a candidate dual CFT is to understand the spectrum of smooth, asymptotically AdS, classical solutions, which are believed to represent semiclassical states in the spectrum of the dual CFT. In this note, we will illustrate a subtlety in the concept of smoothness related to the formulation of the theory in terms of Chern-Simons gauge fields [8],[9]. In this formulation the metric is a derived quantity and in particular there is no natural way to impose its invertibility. As a consequence, many observables that exist in the metric formulation and which involve the inverse metric, are

not natural observables in the Chern-Simons formulation. Instead, the natural observables in the Chern-Simons formulation are based on holonomies of the gauge field, and we will see that these provide a somewhat cruder measure of smoothness.

The solutions we will study are labelled by an integer  $s$  greater than one and correspond to metrics with a conical singularity of excess angle  $2\pi(s - 1)$ . Conical defects in 2+1 dimensional gravity have been extensively studied following the pioneering work [10],[11]. The conical excess solutions were, as far as we know, first explored in [12] as BPS solutions of 3-dimensional supergravity, and were subsequently studied in [13],[14]. Despite being metrically singular, we will argue that the conical excess solutions appear smooth to the natural observables in the Chern-Simons formulation. One new insight coming from our analysis is that requiring the solutions to appear smooth to the point particle probes considered in [15],[16] picks out the conical spaces without angular momentum.

Aside from the singularity in the metric, another reason that conical excesses are usually discarded is that they have energies below that of the global AdS solution, implying that they cannot be part of the spectrum of a unitary dual CFT. Nevertheless, many consistent nonunitary 2-dimensional CFTs are known to exist, some of which have gravity duals<sup>1</sup>, and one might expect the conical spaces to play a role in this context. Indeed, the conical excesses are the pure gravity avatars of similar solutions in higher spin gravity theories<sup>2</sup>, which were argued to have a consistent dual interpretation in a nonunitary semiclassical limit of the dual CFT[19],[20],[21],[22]. The arguments leading to this identification were so far made in the classical approximation in the bulk, and the main motivation of this work was to develop a framework to include bulk quantum corrections in the simplified setting of pure gravity.

We will see that the conical solutions in pure gravity are topological solitons characterized by a winding number. To discuss some of their quantum aspects we will follow the standard approach for quantizing solitons, namely to quantize the fluctuations around the classical solution (see [23] for a review and further references). In the case of pure three dimensional gravity, the fluctuations around a given solution come from acting on it with asymptotic Virasoro generators, or in other words from dressing the solution with boundary graviton excitations. Exponentiating these infinitesimal variations to yield an action of the Virasoro group on the solution, one obtains what is called a coadjoint orbit of the Virasoro group. Each orbit carries a natural Poisson bracket, which upon quantization should lead to a Virasoro representation. The Virasoro coadjoint orbits were classified in [24],[25] and further analyzed by Witten [26]. Subsequent work includes [27],[28],[29], see also [30],[31] for recent discussions of coadjoint orbits in the context of three dimensional

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<sup>1</sup>Known nonunitary examples of holographic duality include the gravity/logarithmic CFT duality (see [17] for a review and further references), and the large  $N$  gauge theories based on a supergroup which were recently discussed in [18].

<sup>2</sup>At first sight, the pure gravity conical spaces may seem metrically more singular than their cousins in higher spin theories, since for the latter one can always apply a higher spin gauge transformation such that the metric has no curvature singularity [19]. However we should keep in mind that metric curvature invariants are not good observables in higher spin theories; for example one can construct higher spin solutions where the metric appears in some gauges to be regular yet the Chern-Simons gauge field describing them is singular.

gravity.

It turns out that the conical spaces give rise to rather special orbits: they are the ‘exceptional’ orbits which possess an  $SO(2, 1)^{(s)} \times SO(2, 1)^{(s)}$  symmetry. Here, the superscript  $s$  indicates that the group is an  $s$ -fold cover of  $SO(2, 1)$ . The generators of this symmetry are embedded in the Virasoro algebra in a different manner for each conical space and, in particular, are different from the symmetry generators of the global AdS solution. By analyzing the exceptional coadjoint orbits one finds that the conical spaces possess boundary graviton fluctuations which lower the Virasoro energy, which is unbounded below [26]. For this reason they cannot be quantized in a way that leads to a unitary highest weight Virasoro representation, and it has so far proved impossible to quantize the exceptional orbits by standard methods.

In this work we will propose a framework to quantize the exceptional orbits which leads to a nonunitary highest weight representation of the Virasoro algebra, constructed as a semiclassical perturbation expansion in the inverse central charge, and proceed to compute the 1-loop correction to the energy of the conical spaces. Furthermore, as anticipated in [26], the quantized exceptional orbits contain a null vector at level  $s$ . Hence the conical spaces constitute an appearance of the most interesting representations of the Virasoro algebra, the degenerate ones, in a gravity context. Our results for the energy correction and the null vector lead us to identify the quantized conical spaces with the degenerate representations of the type  $(1, s)$  in Kac’s classification [32], which are well-known to be nonunitary at large values of the central charge  $c$ . It is fascinating that these representations become unitary, and belong of the spectrum of the Virasoro minimal models, at small values of the central charge  $0 < c < 1$ , which represents the strong coupling regime of the gravity theory.

## 2. Chern-Simons formulation of 3D AdS gravity

Let us briefly review the Chern-Simons formulation of gravity and state our conventions. Pure gravity in 2+1 dimensions with negative cosmological constant can be reformulated as a Chern-Simons theory with gauge group  $SO(1, 2) \times SO(1, 2)$  with opposite levels for the two factors [8],[9]:

$$S = S_{CS}[A] - S_{CS}[\tilde{A}] \quad (2.1)$$

$$S_{CS}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \text{tr}_3 \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \quad (2.2)$$

The gauge potentials  $A, \tilde{A}$  take values in the  $so(1, 2)$  Lie algebra whose generators  $J_a, a = 0, 1, 2$  satisfy the commutation relations<sup>3</sup>

$$[J_a, J_b] = \epsilon_{ab}^{\phantom{ab}c} J_c \quad (2.3)$$

Viewing  $SO(1, 2)$  as the 2+1-dimensional Lorentz group, the generator  $J_0$  is compact and corresponds to spatial rotations, while  $J_{1,2}$  generate boosts. In our conventions the trace

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<sup>3</sup>Indices are raised with the metric  $\eta_{ab} = \text{diag}(-1, 1, 1)$  and  $\epsilon_{012} \equiv 1$ .

in (2.2) is taken in the defining three dimensional representation<sup>4</sup>. The vielbein and spin connection are obtained from

$$\begin{aligned} A &= \left( \omega^a + \frac{e^a}{l} \right) J_a \\ \tilde{A} &= \left( \omega^a - \frac{e^a}{l} \right) J_a \end{aligned} \quad (2.4)$$

where  $\omega^a = \frac{1}{2}\epsilon^a_{bc}\omega^{bc}$  and  $l$  is the AdS radius.

The Chern-Simons field strengths are related to the torsion  $\mathcal{T}^a = de^a + \epsilon^a_{bc}\omega^b \wedge e^c$  and curvature  $\mathcal{R}^a = d\omega^a + \frac{1}{2}\epsilon^a_{bc}\omega^b \wedge \omega^c$  two-forms as follows:

$$\frac{l}{2}(F - \tilde{F}) = \mathcal{T} \quad (2.5)$$

$$\frac{1}{2}(F + \tilde{F}) = \mathcal{R} + \frac{1}{l^2}\epsilon \wedge e. \quad (2.6)$$

Hence the equations of motion, which impose the flatness of  $A$  and  $\tilde{A}$ , imply that the connection  $\omega$  is torsionless and that Einstein's equations hold with negative cosmological constant  $\Lambda = -\frac{2}{l^2}$ .

Writing the action in terms of  $e, \omega$  gives

$$S = \frac{k}{\pi l} \int_{\mathcal{M}} d^3x \sqrt{-g} \left( R + \frac{2}{l^2} \right) + \frac{k}{\pi l} \int_{\partial\mathcal{M}} e^a \wedge \omega_a. \quad (2.7)$$

This allows us to obtain Newton's constant from

$$k = \frac{l}{16G}. \quad (2.8)$$

The Brown-Henneaux central charge [1] is the combination

$$c = \frac{3l}{2G} = 24k. \quad (2.9)$$

The large  $c$  (weak coupling) limit is the semiclassical regime, where the path integral is dominated by classical gravity solutions, while for small  $c$  (strong coupling) quantum corrections are significant.

Gauge transformations act as

$$\begin{aligned} A &\rightarrow \Lambda^{-1} A \Lambda + \Lambda^{-1} d\Lambda \\ \tilde{A} &\rightarrow \tilde{\Lambda}^{-1} \tilde{A} \tilde{\Lambda} + \tilde{\Lambda}^{-1} d\tilde{\Lambda} \end{aligned} \quad (2.10)$$

and in order for  $e^{iS}$  to be gauge-invariant,  $k$  must be quantized in integer units [5], which implies that  $c$  is a multiple of  $24^5$ . By contrast, if we consider Euclidean gravity, the relevant Chern-Simons gauge group is replaced by  $SL(2, \mathbb{C})$  and there is no quantization condition on  $k$  and  $c$  [33] (this is one of the reasons why the relation between Lorentzian and Euclidean Chern-Simons gravity is poorly understood). For most of this paper we will consider the Lorentzian theory but will occasionally comment on the Euclidean case.

<sup>4</sup>Some useful properties are  $\text{tr}_3(J_a J_b) = 2\eta_{ab}$ ,  $\text{tr}_3(J_a J_b J_c) = \epsilon_{abc}$ .

<sup>5</sup>As explained in [5], if we replace the gauge group by an  $n$ -fold cover of  $SO(1, 2)$ ,  $k$  is quantized in units of  $n^{-2}$  and  $c$  is a multiple of  $24/n^2$ . For example taking the gauge group to be  $SL(2, \mathbb{R})$ ,  $c$  should be a multiple of 6.

### 3. Asymptotic symmetries and coadjoint orbits

In this section we will review the analysis of the asymptotic symmetries in the Chern-Simons formulation [34] and rephrase these standard results in the language of coadjoint orbits of the Virasoro group, which we will use in section 5. We start by using the isomorphism  $so(1, 2) \sim sl(2, \mathbb{R})$  to introduce a new Lie algebra basis  $\{V_0, V_{\pm 1}\}$

$$\begin{aligned} V_0 &= -J_2 \\ V_{\pm 1} &= J_0 \pm J_1. \end{aligned} \tag{3.1}$$

which obey  $sl(2, \mathbb{R})$  commutation relations

$$[V_m, V_n] = (m - n)V_{m+n}. \tag{3.2}$$

Note that in the new basis, the compact generator is  $\frac{1}{2}(V_1 + V_{-1})$ . We will take  $\mathcal{M}$  to have the same topology as the global  $AdS_3$  manifold, namely that of the solid cylinder  $\mathbb{R} \times D$ . The time coordinate  $T$  runs along the length of the cylinder while on the disk  $D$  we choose polar coordinates  $\rho, \phi$ , where  $\phi$  has period  $2\pi$ .

Gauge connections  $A, \tilde{A}$  satisfying asymptotically  $AdS$  boundary conditions can be gauge-fixed to the form [34]

$$\begin{aligned} A &= g^{-1}a(x^+)gdx^+ + g^{-1}dg, & g &= e^{\rho V_0} \\ \tilde{A} &= g\tilde{a}(x^-)g^{-1}dx^- + gdg^{-1} \end{aligned} \tag{3.3}$$

where  $x^{\pm} = \phi \pm T$  and  $a(x^+), \tilde{a}(x^-)$  are Lie algebra valued functions which are assumed to be in the so-called highest weight gauge

$$\begin{aligned} a &= V_1 - \frac{12\pi}{c}t(x^+)V_{-1} \\ \tilde{a} &= V_{-1} - \frac{12\pi}{c}\tilde{t}(x^-)V_1. \end{aligned} \tag{3.4}$$

The functions  $t(x^+), \tilde{t}(x^-)$  parameterize the phase space of flat connections obeying asymptotically  $AdS$  boundary conditions. As we will see below, they are to be identified with the left- and right-moving boundary stress tensors. The global  $AdS_3$  solution corresponds to taking

$$t_{AdS} = \tilde{t}_{AdS} = -\frac{c}{48\pi}. \tag{3.5}$$

At fixed time, say  $T = 0$ , the form (3.3) is preserved by gauge parameters of the form

$$\lambda = g^{-1}\epsilon(\phi)g, \quad \tilde{\lambda} = g\tilde{\epsilon}(\phi)g^{-1}. \tag{3.6}$$

under which  $a, \tilde{a}$  transform as

$$\delta a = \epsilon' + [a, \epsilon], \quad \delta \tilde{a} = \tilde{\epsilon}' + [\tilde{a}, \tilde{\epsilon}]. \tag{3.7}$$

Decomposing  $\epsilon, \tilde{\epsilon}$  as

$$\begin{aligned} \epsilon(\phi) &= j(\phi)V_1 + \epsilon_0(\phi)V_0 + \epsilon_{-1}(\phi)V_{-1} \\ \tilde{\epsilon}(\phi) &= \tilde{j}(\phi)V_{-1} + \tilde{\epsilon}_0(\phi)V_0 + \tilde{\epsilon}_1(\phi)V_1 \end{aligned} \tag{3.8}$$

the requirement that the gauge transformations preserve the highest weight gauge (3.4) fixes  $\epsilon_0, \epsilon_{-1}$  and  $\tilde{\epsilon}_0, \tilde{\epsilon}_1$  in terms of  $j$  and  $\tilde{j}$  respectively. Such transformations should be viewed as infinitesimal asymptotic symmetries, which are therefore parameterized by two functions on the circle  $j(\phi)$  and  $\tilde{j}(\phi)$ . Under the action of the infinitesimal asymptotic symmetries  $t(\phi)$  and  $\tilde{t}(\phi)$  transform as

$$\begin{aligned}\delta_j t &= 2tj' + jt' - \frac{c}{24\pi}j''' \\ \delta_{\tilde{j}} \tilde{t} &= 2\tilde{t}\tilde{j}' + \tilde{j}\tilde{t}' - \frac{c}{24\pi}\tilde{j}'''\end{aligned}\tag{3.9}$$

The conserved charge  $q_j$  corresponding to the asymptotic symmetry  $j(\phi)$  is

$$q_j(t) = \int_0^{2\pi} d\phi j(\phi)t(\phi)\tag{3.10}$$

and similarly for the right-moving charges  $\tilde{q}_{\tilde{j}}$  (for the rest of this section, we will display only the formulas in the left-moving sector, the right-moving side proceeding analogously). The conserved charges are linear functionals on the phase space and we would like to compute their Poisson brackets. These can be deduced from the fact for every function  $\mathcal{O}$  on the phase space, its variation under  $j$  arises from the Poisson bracket with  $q_j$ :  $\{q_j, \mathcal{O}\}_{PB} = \delta_j \mathcal{O}$ . Taking  $\mathcal{O}$  to be the conserved charge  $\mathcal{O} = q_k$  we find the Poisson bracket

$$\{q_j, q_k\}_{PB}(t) = \int_0^{2\pi} d\phi k \delta_j t\tag{3.11}$$

$$= q_{(j'k - jk')}(t) - \frac{c}{48\pi} \int_0^{2\pi} d\phi (j'''k - jk''')\tag{3.12}$$

Let us review the group theoretic meaning of this expression. If  $c$  were zero, this Poisson bracket would realize the Lie algebra  $\text{diff}(S^1)$  of reparametrizations of the circle in the sense that

$$\{q_j, q_k\}_{PB} = q_{[k,j]}\tag{3.13}$$

where  $[k, j] = j'k - jk'$  is the commutator in  $\text{diff}(S^1)$ . Therefore  $j(\phi), k(\phi)$  should be thought of as components of tangent vectors  $j(\phi)\partial_\phi, k(\phi)\partial_\phi$ . As for  $t(\phi)$ , the expression for the charge (3.10) allows us to identify  $t(\phi)$  as an element of the vector space dual to  $\text{diff}(S^1)$ . The transformation law (3.9) at  $c = 0$  defines the coadjoint representation of  $\text{diff}(S^1)$  and shows that  $t(\phi)$  should be seen as the component of a quadratic differential  $t(\phi)d\phi^2$ .

When  $c$  is nonzero, the Poisson bracket (3.12) instead realizes a central extension of  $\text{diff}(S^1)$ , the Virasoro algebra, as we shall presently review. Extending  $\text{diff}(S^1)$  with a central generator  $\hat{c}$ , elements of the Virasoro algebra can be represented as pairs

$$(j(\phi), n) \longleftrightarrow j(\phi)\partial_\phi + n\hat{c}\tag{3.14}$$

where  $n$  is a real number. The commutation relations are

$$[(j, n), (k, m)] = \left( (jk' - j'k), \frac{1}{48\pi} \int_0^{2\pi} d\phi (j'''k - jk''') \right).\tag{3.15}$$

Similarly, we extend the dual vector space by including the constant parameter  $c$  as an extra coordinate. The dual vector space to the Virasoro algebra consists of pairs of the form  $(t(\phi), c)$ , and the pairing between adjoint and coadjoint vectors is given by the generalization of the expression (3.10) for the charge:

$$q_{(j,n)}(t, c) = \int_0^{2\pi} d\phi j t + c n \equiv \langle (t, c), (j, n) \rangle \quad (3.16)$$

The transformation law (3.9) can be extended in such a way that the pairing is Virasoro-invariant, meaning

$$\langle \delta_{(k,m)}(t, c), (j, n) \rangle + \langle (t, c), [(k, m), (j, n)] \rangle = 0. \quad (3.17)$$

This leads to

$$\delta_{(j,n)}(t, c) = (2tj' + jt' - \frac{c}{24\pi} j''', 0). \quad (3.18)$$

This transformation law defines the coadjoint representation of the Virasoro algebra. Using these definitions one finds that the Poisson bracket (3.12) can be written as

$$\{q_{(j,n)}, q_{(k,m)}\}_{PB}(t, c) = q_{[(k,m), (j,n)]}(t, c) \quad (3.19)$$

from which we see that the Poisson bracket (3.12) indeed realizes the Virasoro algebra.

The Poisson bracket can be written in a more standard form by choosing the following basis for the charges generating  $\text{diff}(S^1)$ :

$$l_m = q_{(e^{im\phi}, 0)}. \quad (3.20)$$

In this basis the Poisson brackets take the form

$$-i\{l_m, l_n\}_{PB} = (m - n)l_{m+n} + \frac{\hat{c}m^3}{12}\delta_{m,-n}. \quad (3.21)$$

Note that our Virasoro energies  $l_0, \tilde{l}_0$  are naturally defined on the boundary cylinder, and on the global AdS solution (which will turn out to correspond to the  $SL(2, \mathbb{R})$  invariant vacuum in the dual CFT) they take the values

$$l_0((t_{AdS}, c)) = \tilde{l}_0((\tilde{t}_{AdS}, c)) = -\frac{c}{24}. \quad (3.22)$$

This corresponds to the Casimir energy of a cylinder of circumference  $2\pi$ . To make contact with the more standard conventions where  $l_0$  and  $\tilde{l}_0$  act on the plane one should shift them by  $\frac{c}{24}$ .

Extending this analysis to include the right-moving sector, we conclude that the asymptotic charges generate two copies of the Virasoro algebra through Poisson brackets. The combinations  $(t(\phi), c)$  and  $(\tilde{t}(\phi), c)$  transform in the coadjoint representation of the respective Virasoro algebras. At fixed central charge  $c$ , asymptotically AdS solutions come in families obtained by acting on a given solution with the Virasoro symmetries, which are referred to as coadjoint orbits of the Virasoro group. Physically, moving around on a coadjoint orbit can be seen as dressing (or undressing) a given solution with boundary



graviton excitations. A standard result, which we will use to quantize coadjoint orbits in section 5, is that each orbit possesses a natural symplectic form and hence a Poisson bracket (the Kirillov-Kostant bracket) such that the restrictions of the charges to the orbit obey (3.19).

Of course, not all coadjoint orbits are physically acceptable, as some of them may correspond to singular gravity solutions, and the issue of regularity will be the subject of the next section. The regular coadjoint orbits are expected to correspond to semiclassical states in the quantum theory, and upon quantizing them we expect to obtain Virasoro representations belonging to the spectrum of the dual CFT.

## 4. Conical spaces as topological Chern-Simons solitons

In this section we will revisit, in the pure gravity context, the conical solutions which were recently studied in higher spin gravity [19]. We will emphasize their similarity to solitons in the sense that they carry a topological winding number, and discuss their smoothness as seen by observables both in the metric and Chern-Simons formulations.

### 4.1 Winding numbers

Let us study the space of asymptotically AdS solutions on the solid cylinder in more detail. At any fixed time  $T$ , the flat connections  $a, \tilde{a}$  (see (3.4)) on the boundary cylinder can be expressed as

$$a(\phi)d\phi = h^{-1}dh, \quad \tilde{a}(\phi)d\phi = \tilde{h}^{-1}d\tilde{h} \quad (4.1)$$

where  $h(\phi), \tilde{h}(\phi)$  are maps from the boundary circle into the gauge group  $SO(1, 2)$ . These are classified by the first homotopy group of  $SO(1, 2)$ , and since  $SO(1, 2)$  is contractible to its maximal compact subgroup  $U(1)$ , the first homotopy group is  $\pi_1(SO(1, 2)) = \mathbb{Z}$ . Hence the space of flat connections on the boundary consists of topological sectors labelled by the two winding numbers  $s, \tilde{s}$  of the maps  $h(\phi), \tilde{h}(\phi)$ , which measure how many times the  $U(1)$  directions in  $SO(1, 2) \times SO(1, 2)$  are traversed when we go around the boundary circle. The following are representative maps with winding numbers  $s, \tilde{s}$

$$\begin{aligned} m_s(\phi) &= e^{\frac{s\phi}{2}(V_1 + V_{-1})} \\ \tilde{m}_{\tilde{s}}(\phi) &= e^{\frac{\tilde{s}\phi}{2}(\tilde{V}_{-1} + \tilde{V}_1)}. \end{aligned} \quad (4.2)$$

These however don't give rise, upon substituting in (4.1), to connections in the highest weight gauge (3.4). This can be remedied by choosing different representatives related by a constant gauge transformation

$$\begin{aligned} h_s(\phi) &= b m_s(\phi) b^{-1} = e^{\phi(V_1 + \frac{s^2}{4}V_{-1})}, & b &= e^{\ln \frac{s}{2}V_0} \\ \tilde{h}_{\tilde{s}}(\phi) &= \tilde{b} \tilde{m}_{\tilde{s}}(\phi) \tilde{b}^{-1} = e^{\phi(V_{-1} + \frac{\tilde{s}^2}{4}V_1)}, & \tilde{b} &= e^{-\ln \frac{\tilde{s}}{2}V_0}. \end{aligned} \quad (4.3)$$

Substituting in (4.1) gives the following highest weight gauge connections<sup>6</sup>

$$\begin{aligned} a_s &= V_1 + \frac{s^2}{4} V_{-1} \\ \tilde{a}_{\tilde{s}} &= V_{-1} + \frac{\tilde{s}^2}{4} V_1. \end{aligned} \quad (4.4)$$

We will label these solutions by their winding numbers  $(s, \tilde{s})$  in what follows. The  $(s, \tilde{s})$  solution is characterized by constant covectors  $(t_s, c)$  and  $(\tilde{t}_{\tilde{s}}, c)$  with

$$t_s = -\frac{cs^2}{48\pi}, \quad \tilde{t}_{\tilde{s}} = -\frac{c\tilde{s}^2}{48\pi}. \quad (4.5)$$

From (3.16) we read off the Virasoro energies

$$l_0((t_s, c)) = -\frac{cs^2}{24}, \quad \tilde{l}_0((\tilde{t}_{\tilde{s}}, c)) = -\frac{c\tilde{s}^2}{24}. \quad (4.6)$$

The other Virasoro charges vanish on these solutions. Comparing to (3.5), we see that the case  $s = \tilde{s} = 1$  corresponds to global AdS: it is a somewhat peculiar feature of the Chern-Simons description that the natural classical vacuum of the theory appears in a winding sector. We note that the winding states with  $s, \tilde{s} > 1$  have energies below the AdS vacuum energy. Since in a unitary CFT all primaries have conformal weights above the  $SL(2, \mathbb{R})$  invariant vacuum, this is already an indication that these winding states can only play a role in nonunitary versions of holography, as we shall see in more detail below.

## 4.2 Metric-like observables

Let us now discuss whether the  $(s, \tilde{s})$  winding solutions are smooth. Since the  $\phi$ -circles (i.e. the curves of constant  $T, \rho$ ) are by assumption contractible, our coordinate system is singular at some value of  $\rho$  (which we call the ‘origin’), where the  $\phi$  coordinate is ill-defined. Our solutions satisfy the equations of motion everywhere except possibly in the origin. To decide whether the solution is singular in the origin, we will look at suitable gauge-invariant observables which could measure such a singularity. As we already mentioned in the Introduction, the analysis depends on whether we work in the Chern-Simons formulation, where the natural observables come from holonomies of the gauge field, or the metric formulation, where we have at our disposal the standard curvature invariants which are constructed using the inverse metric. Let’s start by addressing smoothness in the metric formulation.

For simplicity we restrict our attention to the solutions where  $\tilde{s} = s$  which have vanishing angular momentum<sup>7</sup>. The corresponding metrics are, after a shift  $\rho \rightarrow \rho + \ln \frac{s}{2}$ ,

$$ds_{(s,s)}^2 = l^2 \left[ -s^2 \cosh^2 \rho dT^2 + d\rho^2 + s^2 \sinh^2 \rho d\phi^2 \right] \quad (4.7)$$

<sup>6</sup>Here we encounter the subtlety that (4.3) is a good gauge transformation only when both  $s$  and  $\tilde{s}$  are positive. Since we can flip the sign of both  $s$  and  $\tilde{s}$  by sending  $\phi \rightarrow -\phi$ , we can only transform the maps with  $s\tilde{s} > 0$  to the highest weight gauge. We will assume from now on that  $s$  and  $\tilde{s}$  are positive.

<sup>7</sup>When  $s \neq \tilde{s}$ , the metric is that of a spinning conical defect:

$$ds_{(s,\tilde{s})}^2 = l^2 \left[ d\rho^2 + \frac{s\tilde{s}}{2} \cosh 2\rho dx^+ dx^- - \frac{s^2}{4} (dx^+)^2 - \frac{\tilde{s}^2}{4} (dx^-)^2 \right]$$

which apart from a curvature singularity also contains closed timelike curves.

As anticipated in (3.5), for  $s = 1$  we recover the standard global AdS metric. In the limit  $s \rightarrow 0$  (where we cannot perform the shift of the  $\rho$ -variable) the metric is that of the zero mass, zero angular momentum BTZ black hole

$$ds_{(0,0)}^2 = l^2 [d\rho^2 + e^{2\rho} (-dt^2 + d\phi^2)] \quad (4.8)$$

which has energy above that of global AdS.

For  $s > 1$  the metric (4.7) can be seen to have a curvature singularity in  $\rho = 0$ , corresponding to a conical singularity with an excess angle of  $2\pi(s - 1)$  [10],[11]. Let's rederive this fact in a way that emphasizes the difference between the metric and Chern-Simons formulations. In the metric formulation we postulate that the vielbein is invertible. If this is the case, we can construct many gauge-invariant observables besides the eigenvalues of the holonomies (4.16). For example, for every (non-null) two-surface we can define a surface observable

$$\mathcal{O}(\mathcal{S}) = \int_{\mathcal{S}} (F^a + \tilde{F}^a) e_a^\mu n_\mu \quad (4.9)$$

where  $n_\mu$  is the unit normal to the surface. Now let's evaluate  $\mathcal{O}(\mathcal{D})$  with  $\mathcal{D}$  a disc  $0 \leq \rho \leq \rho_0$  at constant  $T$ , in the background of the  $(s, s)$  solution. For any solution with constant  $t = \tilde{t}$  we can rewrite  $\mathcal{O}(\mathcal{D})$  as

$$\mathcal{O}(\mathcal{D}) = \int_{\mathcal{D}} \left( \tilde{R} + \frac{2}{l^2} \right) \sqrt{\tilde{g}} d\rho d\phi \quad (4.10)$$

with  $\tilde{g}_{\mu\nu}$  the induced metric on  $\mathcal{D}$  and  $\tilde{R}$  its scalar curvature. Due to the equations of motion the integrand vanishes everywhere, except possibly in the origin, and we can replace (4.10) by

$$\mathcal{O}(\mathcal{D}) = \int_{\mathcal{D}_\epsilon} \sqrt{\tilde{g}} \tilde{R} d\rho d\phi \quad (4.11)$$

where  $\mathcal{D}_\epsilon$  is a tiny disc around the origin. The spatial metric near the origin is

$$d\tilde{s}^2 \sim l^2 [d\rho^2 + s^2 \rho^2 d\phi^2] \quad (4.12)$$

which can be written as the flat metric  $l^2 du d\bar{u}$  in terms of the complex coordinate  $u = \rho e^{is\phi}$ . However, the transformation from  $(\rho, \phi)$  and  $(u, \bar{u})$  is not one-to-one, rather it is one-to-one for the coordinate  $v = u^{1/s}$ , in terms of which the metric is only conformally flat, with a conformal factor which is singular in the origin:

$$d\tilde{s}^2 \sim l^2 |v|^{2(s-1)} dv d\bar{v}. \quad (4.13)$$

Hence a useful way to picture the geometry near the origin is as the Riemann surface of the function  $u^{1/s}$ , i.e. as an  $s$ -sheeted branched covering over the complex plane. We then evaluate

$$\sqrt{\tilde{g}} \tilde{R} = (1 - s) \partial_v \partial_{\bar{v}} \ln |v|^2 = 4\pi(1 - s) \delta^{(2)}(v, \bar{v}) \quad (4.14)$$

from which we find

$$\mathcal{O}(\mathcal{D}) = 4\pi(1 - s). \quad (4.15)$$

Hence from the metric point of view only the global AdS solution with  $s = 1$  is regular while the remaining winding states are not. The surface observable  $\mathcal{O}(\mathcal{D})$  which sees the singularity is not a natural observable in the Chern-Simons formulation of the theory as it explicitly involves the inverse vielbein. From the Chern-Simons point of view, the natural observables are rather based on the holonomies of the gauge field which we shall now discuss.

### 4.3 Holonomies

The natural observables in a three-dimensional Chern-Simons theory are related to the holonomies of the gauge fields around a closed curve  $\mathcal{C}$ :

$$H(\mathcal{C}) = \mathcal{P}e^{\int_{\mathcal{C}} A}, \quad \tilde{H}(\mathcal{C}) = \mathcal{P}e^{\int_{\mathcal{C}} \tilde{A}} \quad (4.16)$$

The eigenvalues of  $H(\mathcal{C}), \tilde{H}(\mathcal{C})$  are gauge-invariant and, when the connections  $A, \tilde{A}$  are flat, depend only on the homotopy class of  $\mathcal{C}$ . However, they do depend on the representation of the gauge group used to evaluate them. Since we are considering the pure gravity theory, which contains only the gauge fields in the adjoint representation, it's natural to evaluate the holonomies (4.16) in the adjoint representation, in our case the 3-dimensional representation of  $SO(1,2)$ . For a smooth flat gauge connection, the holonomy around a contractible loop should be unity. Here we shall take  $\mathcal{C}$  to be a  $\phi$ -circle, which is by assumption contractible, in the  $(s, \tilde{s})$  winding solution. We find that

$$\begin{aligned} H &\sim e^{2\pi a} \sim e^{-2\pi i s V_0} \\ \tilde{H} &\sim e^{2\pi \tilde{a}} \sim e^{-2\pi i \tilde{s} V_0} \end{aligned} \quad (4.17)$$

where in the last step we have used

$$\frac{1}{2}(V_1 + V_{-1}) = M^{-1}(-iV_0)M \quad (4.18)$$

with  $M = e^{-i\pi/4(V_1 - V_{-1})}$ . Since the eigenvalues of  $V_0$  in the adjoint representation are 1, 0 and  $-1$ , we see that the holonomy is indeed trivial and all the  $(s, \tilde{s})$  winding solutions appear to be smooth [13]. Note that, if we replace the adjoint representation with any other finite-dimensional representation of  $SO(2,1)$ , the holonomy remains trivial as the eigenvalues of  $V_0$  are integer in these representations. Hence from the point of view of these observables, all the  $(s, \tilde{s})$  solutions appear to be regular.

More generally, we can ask if the  $(s, \tilde{s})$  winding states still appear smooth if we probe them with external matter, for example a spinning point particle probe of mass  $M$  and spin  $J$ . It was argued in [15] that such probes are related to holonomies (4.16) evaluated in infinite-dimensional unitary representations of the gauge group. We shall follow here the recent discussion<sup>8</sup> [16] (see [37] for the extension to spinning particles) to which we refer for more details. We will focus on the gauge-invariant Wilson loop operator

$$W_R(\mathcal{C}) = \text{tr}_R \left( \mathcal{P}e^{\int_{\mathcal{C}} (A + \tilde{A})} \right). \quad (4.19)$$

---

<sup>8</sup>See also [35] for a closely related description of point particles in the Chern-Simons formulation, and [36] for the precise relation between the two descriptions.

where  $\mathcal{C}$  is a closed loop and  $R$  is a representation of the gauge group. We take  $R$  to be the infinite-dimensional highest weight representation of  $SO(1, 2) \times SO(1, 2)$  built on a primary state  $|h, \tilde{h}\rangle$ :

$$\begin{aligned} L_1|h, \tilde{h}\rangle &= \tilde{L}_1|h, \tilde{h}\rangle = 0 \\ L_0|h, \tilde{h}\rangle &= h|h, \tilde{h}\rangle, \quad \tilde{L}_0|h, \tilde{h}\rangle = \tilde{h}|h, \tilde{h}\rangle \end{aligned} \quad (4.20)$$

The quantum numbers  $h, \tilde{h}$  are related to the mass and spin of the probe as

$$lM = h + \tilde{h}, \quad J = h - \tilde{h}. \quad (4.21)$$

The physical meaning of this Wilson loop was found in [36] to be as follows. When  $\mathcal{C}$  is noncontractible, the Wilson loop is related to the proper distance along the curve (or a generalization thereof for spinning particles). For example, for  $J = 0$  and  $\mathcal{C}$  the  $\phi$ -circle in the BTZ black hole background,  $W_R(\mathcal{C})$  measures the Bekenstein-Hawking entropy. When  $\mathcal{C}$  is contractible, it measures rather the phase picked up by the wavefunction of the test particle<sup>9</sup> when going around  $\mathcal{C}$ , and when this phase is nontrivial it measures a singularity in the gauge field as seen by the probe. For example, for the  $\phi$ -circle in the global AdS background one finds a trivial phase, while for conical defect solutions with  $-c/48\pi < t, \tilde{t} < 0$  the phase is nontrivial, signaling a singularity.

Let us now compute the Wilson loop in the background of our conical excess solutions labelled by  $(s, \tilde{s})$ , where we take  $\mathcal{C}$  to be a  $\phi$ -circle. The Wilson loop can be evaluated by representing the trace as a path integral over auxiliary variables, see Appendix E in [37] for details in the case of spinning particles. For large  $h, \tilde{h}$ , where the point particle approximation is valid, the auxiliary path integral can be approximated by a saddle point contribution with the result

$$W_R(\mathcal{C}) = e^{-\pi \text{tr}_3((h\lambda_\phi - \tilde{h}\tilde{\lambda}_\phi)V_0)} \quad (4.22)$$

where the trace is taken in the 3-dimensional representation of  $SO(1, 2)$ . The matrices  $\lambda_\phi, \tilde{\lambda}_\phi$  are the eigenvalue matrices of  $a_\phi$  and  $\tilde{a}_\phi$  respectively, which we already determined in (4.17):

$$\lambda_\phi = -isV_0, \quad \tilde{\lambda}_\phi = -i\tilde{s}V_0. \quad (4.23)$$

Substituting in (4.22) we obtain

$$\boxed{W_R(\mathcal{C}) = e^{i\pi((s-\tilde{s})M + (s+\tilde{s})J)}. \quad (4.24}$$

Let us discuss this result first for the case where the probe particles are bosons, so that  $J$  is an integer, with arbitrary mass  $M$ . From (4.24) we see that their wavefunctions are single-valued only when

$$s = \tilde{s} \quad (4.25)$$

---

<sup>9</sup>An argument for this goes as follows. Let's take  $\mathcal{C}$  to be a  $\phi$ -circle and consider the wavefunction of a position eigenstate, i.e. the propagator  $\langle \rho, \phi, T | \rho', \phi', T' \rangle$ , which can be represented as a sum over paths weighted by (4.19). When sending  $\phi \rightarrow \phi + 2\pi$ , each contributing path  $\mathcal{P}$  gets deformed to a new path homotopic to  $\mathcal{P} + \mathcal{C}$ . Hence the phase picked up by the propagator is given by the Wilson loop around  $\mathcal{C}$ .

i.e. the left- and right winding numbers must be equal, which restricts to the non-spinning defects with  $l_0 = \tilde{l}_0$ . For fermionic probe particles,  $J$  is half-integer and in a regular background the wavefunction should pick up a phase  $-1$  when going around the origin. From (4.24) we see that only the conical spaces with odd  $s$  appear regular to fermionic probes, which obey the familiar Neveu-Schwarz boundary conditions on the boundary cylinder. The solutions with even  $s$  do appear singular to fermionic probes, which experience the insertion of a worldline defect which causes them to obey Ramond boundary conditions on the boundary cylinder. For  $s = 0$  this reduces to the familiar result that the zero-mass BTZ black hole lives in the Ramond sector of the dual CFT. Since we are in 2+1 dimensions we can also consider probes with fractional spin,  $J \in \mathbb{N} + 1/n$ , with  $n$  a positive integer, whose wavefunctions should pick up a phase  $e^{\frac{2\pi i}{n}}$ . From (4.24) we see that only the conical spaces with  $s = 1 \bmod n$  appear regular to such probes.

To summarize, we have found that from the point of view of natural Chern-Simons observables all  $(s, \tilde{s})$  conical solutions appear regular in pure gravity, while adding bosonic probes selects the non-spinning solutions with  $s = \tilde{s}$ . Adding fermionic or fractional spin matter further restricts the allowed values of  $s$ .

We end this section by commenting on the fate of the conical spaces when working in Euclidean rather than Lorentzian signature. In this case, the Chern-Simons gauge group is to be replaced by  $SL(2, \mathbb{C})$ , which has the property that all closed loops are contractible,  $\pi_1(SL(2, \mathbb{C})) = 0$ . In this case, the analytic continuation of the conical solutions still yields smooth Euclidean solutions which however don't carry any topological winding charge. In fact, independent of the signature, the search for smooth solutions which lie on the orbits of a constant covector yields precisely the  $(s, \tilde{s})$  solutions and nothing else [19]. We should note that there also exist coadjoint orbits which do not contain a constant covector [26], whose role in 3D gravity remains to be fully understood (see [30] for a further discussion of these solutions).

## 5. Quantizing the conical spaces

In the previous section we have provided some plausibility arguments that the conical spaces are soliton-like solutions which appear smooth to observables in the Chern-Simons formulation, and we would now like to quantize them and identify them with states in a dual CFT. The standard method to quantize solitons is to make a perturbative expansion of the action in small fluctuations around the soliton and proceed to quantize them. This gives the correction to the energy of the solution and the spectrum of excited bound states in an expansion in  $\hbar$  (or, in our case,  $1/c$ ) [23].

Since the Chern-Simons theory is topological all on-shell fluctuations arise from gauge transformations (2.10) of  $A, \tilde{A}$ . However as we reviewed in section 3, after imposing asymptotically AdS boundary conditions the subset of transformations (3.8) should rather be seen as generators of global symmetries which change the solution. Therefore the fluctuations we are to quantize come from acting on our solutions with asymptotic symmetries, i.e. from displacements on the coadjoint orbits of the solutions. In order to quantize them we use the fact that each coadjoint orbit carries a well-defined Poisson bracket, the Kirillov-Kostant

bracket, under which the Poisson brackets of the Virasoro charges on the orbit satisfy the classical Virasoro algebra (3.21).

Upon quantizing this Poisson bracket we expect to obtain representations of the Virasoro group<sup>10</sup> which we would like to determine<sup>11</sup>. As we shall see, the main obstacle in carrying out this programme is that the conical orbits have negative directions for the energy  $l_0$ . For this reason it has so far proved impossible to quantize these orbits by standard methods. We will here propose a semiclassical quantization of the conical orbits which leads to non-unitary highest weight Virasoro representations. The most important property of the resulting representation is that it will turn out to have a null vector at level  $s$ , and we will make the connection with Kac's classification of degenerate representations.

### 5.1 Symplectic form on coadjoint orbits

We start by briefly reviewing the Kirillov-Kostant symplectic structure on the Virasoro coadjoint orbits, referring to [26], [27] for more details. It will be convenient to parameterize points on the coadjoint orbit of some fixed covector  $(t(\phi), c)$  by the finite group element which maps  $(t(\phi), c)$  to the desired point<sup>12</sup>. In our case the group which acts on the orbits is  $Diff(S^1)$ , the group of diffeomorphisms of the circle, i.e. periodic maps

$$\phi \mapsto F(\phi), \quad F(\phi + 2\pi) = F(\phi) + 2\pi. \quad (5.1)$$

The infinitesimal coadjoint action (3.18) integrates to the well-known finite transformation which involves the Schwarzian derivative  $S(F)$ :

$$(t(\phi), c) \xrightarrow{F} (t_F(\phi), c) \quad (5.2)$$

$$t_F(\phi) = t(F(\phi))(F')^2 - \frac{c}{24\pi} S(F) \quad (5.3)$$

$$S(F) \equiv \frac{F'''}{F'} - \frac{3}{2} \left( \frac{F''}{F'} \right)^2 \quad (5.4)$$

Combining this with (3.16) we find the Virasoro charges along a coadjoint orbit

$$l_n((t_F, c)) = \int_0^{2\pi} e^{in\phi} \left( t(F(\phi))(F')^2 - \frac{c}{24\pi} S(F) \right). \quad (5.5)$$

The Kirillov-Kostant symplectic form at the point  $(t_F, c)$  on the orbit of the covector  $(t, c)$  can be written as [27]

$$\Omega = - \left\langle (t_F, c), \left[ \left( \frac{\delta F}{F'}, 0 \right), \left( \frac{\delta F}{F'}, 0 \right) \right] \right\rangle \quad (5.6)$$

$$= \int_0^{2\pi} d\phi \left[ \left( t_0(F) - \frac{c}{24(F')^2} S(F) \right) \delta F' \wedge \delta F + \frac{c}{48\pi} \left( \frac{\delta F}{F'} \right)''' \wedge \frac{\delta F}{F'} \right] \quad (5.7)$$

<sup>10</sup>See [38] for a general perspective on coadjoint orbits and representation theory.

<sup>11</sup>To be more precise, since our setup will be perturbative around the conical solutions, we will obtain representations of the Virasoro algebra rather than the Virasoro group.

<sup>12</sup>Once again we focus in this section on the left-moving sector, the right-moving sector being analogous.

where the brackets in the first line denote the pairing introduced in (3.16). We have here introduced the notation  $\delta$  to denote the de Rham operator acting on phase space variables, as opposed to the spacetime de Rham operator  $d$ . The main property of this symplectic form is that the Poisson brackets of the charges along the orbit (5.5) are guaranteed to satisfy the classical Virasoro algebra (3.21). Following [27], (5.7) can be rewritten in a useful simplified form as a total derivative

$$\Omega = \int_0^{2\pi} d\phi \delta \left( t(F(\phi)) F' \delta F - \frac{c}{48\pi} \frac{\delta F}{F'} \left( \frac{F'''}{F'} - 2 \left( \frac{F''}{F'} \right)^2 \right) \right). \quad (5.8)$$

## 5.2 Conical spaces and exceptional orbits

Next we want to apply these general formulas to the coadjoint orbits of the conical solutions. We recall that the conical orbits are those of the constant covectors  $(t_s, c)$ , where

$$t_s = -\frac{cs^2}{48\pi}. \quad (5.9)$$

and  $s > 1$ .

Let us first analyze the symmetries of the conical orbits. Any covector is trivially left invariant by  $\hat{c} = (0, 1)$ , and from (3.18) we see that constant covectors are also invariant under the action of  $l_0$ . The conical covectors are special in that they are left invariant by two more generators; indeed one easily checks from (3.18) that

$$\delta_{l_s}((t_s, c)) = \delta_{l_{-s}}((t_s, c)) = 0. \quad (5.10)$$

The infinitesimal transformations generated by the charges  $l_0, l_{\pm s}$ <sup>13</sup> integrate to finite reparameterizations of the circle of the form [29]

$$F(\phi) = \frac{1}{is} \ln \frac{\alpha e^{is\phi} + \beta}{\bar{\beta} e^{is\phi} + \bar{\alpha}}, \quad |\alpha|^2 - |\beta|^2 = 1. \quad (5.11)$$

These form an  $s$ -fold cover of  $SO(2, 1)$  which we denote as  $SO(2, 1)^{(s)}$ . Hence the conical orbits have the structure of the coset spaces  $Diff(S^1)/SO(2, 1)^{(s)}$  and are often referred to as exceptional coadjoint orbits.

Next we would like to compute the Virasoro charges along the conical orbits. We parameterize the elements of  $Diff(S^1)$  as

$$F(\phi) = \phi + \sum_{n \in \mathbb{Z}} f_n e^{-in\phi}, \quad (5.12)$$

and reality of  $F(\phi)$  implies that

$$f_m^* = f_{-m}. \quad (5.13)$$

In view of the symmetries discussed above, the  $f_n$  for  $n \neq 0, s, -s$  are coordinates on the orbit of  $(t_s, c)$  (it will be a good check to see that the Virasoro charges are independent of  $f_0, f_s$  and  $f_{-s}$ ). We start with the expression for the energy  $l_0$ , which using (5.5) reads

$$l_0((t_{s,F}, c)) = -\frac{cs^2}{24} + \frac{c}{12} \sum_{m \in \mathbb{N}} m^2 (m^2 - s^2) |f_m|^2 + \mathcal{O}(f^3) \quad (5.14)$$

---

<sup>13</sup>This symmetry algebra was referred to as a ‘twisted  $sl(2, \mathbb{R})$ ’ in [20].



It's immediately clear that there is a major difference between the  $s = 1$  and  $s > 1$  cases: from (5.14) we see that for  $s = 1$  the covector  $(t_s, c)$  around which we are expanding is a minimum of the energy, while for  $s > 1$  it is only a saddle point. The fluctuations  $f_m$  with  $|m| < s$  represent unstable directions of the conical spaces. Moreover, it can be shown that the energy on the conical orbits is unbounded below [26].

The other Virasoro charges along the conical orbits are<sup>14</sup>

$$l_m((t_{s,F}, c)) = \frac{ic}{12} m(s^2 - m^2) f_m + \mathcal{O}(f^2); \quad m \neq 0, s, -s \quad (5.17)$$

$$l_{\pm s}((t_{s,F}, c)) = \frac{c}{24} \sum_{m \in \mathbb{Z}} (m^2 - s^2)((m \mp s)^2 - s^2) f_m f_{\pm s - m} + \mathcal{O}(f^3). \quad (5.18)$$

Next we evaluate the expression (5.8) for the symplectic form:

$$\Omega = -\frac{ic}{12} \sum_{m \in \mathbb{N}} m(m^2 - s^2) \delta f_m \wedge \delta f_{-m} + \dots \quad (5.19)$$

where the dots mean that we have omitted terms of higher order in the  $f_n$ .

### 5.3 Semiclassical expansion

Now we would like to use the symplectic form (5.19) to quantize the conical orbits. We start by introducing new coordinates  $a_m, m \neq 0, s, -s$ , on the orbit:

$$a_m = \sqrt{\frac{c}{12}} f_m + \mathcal{O}(f^2) \quad (5.20)$$

in terms of which the symplectic form (5.8) is

$$\Omega = -i \sum_{m \in \mathbb{N}} m(m^2 - s^2) \delta a_m \wedge \delta a_{-m}. \quad (5.21)$$

Here, the  $\mathcal{O}(f^2)$  terms in (5.20) are chosen such that (5.21) is exact without further higher order corrections; it follows from Darboux's theorem<sup>15</sup> that this is indeed possible. Note that the reality of  $\Omega$  implies that

$$a_m^* = a_{-m}, \quad (5.22)$$

which at leading order order follows from (5.13). The  $a_m$  are related to canonical coordinates  $x_m, p_m$ , in terms of which  $\Omega = \sum_{m \in \mathbb{N}_0 \setminus \{s\}} dp_m \wedge dx_m$ , as follows:

$$a_{\pm m} = \frac{1}{\sqrt{2|m(m^2 - s^2)|}} (x_m \mp ip_m), \quad \text{for } 0 < m < s \quad (5.23)$$

$$a_{\pm m} = \frac{1}{\sqrt{2|m(m^2 - s^2)|}} (x_m \pm ip_m), \quad \text{for } m > s. \quad (5.24)$$

---

<sup>14</sup>The second expression requires some explanation. Applying (5.5) one obtains

$$l_{\pm s}((t_{s,F}, c)) = \frac{c}{24} \sum_{m \in \mathbb{Z}} m(m \mp s)(m^2 \pm ms - 3s^2) f_m f_{\pm s - m} + \mathcal{O}(f^3). \quad (5.15)$$

One can check this expression does not depend on  $f_0, f_s$  and  $f_{-s}$ , which can be made more explicit by adding zero in the form

$$0 = \mp \frac{c}{24} \sum_{m \in \mathbb{Z}} m(m \mp s) \left( m \mp \frac{s}{2} \right) s f_m f_{\pm s - m} \quad (5.16)$$

after which one obtains (5.18).

<sup>15</sup>Or rather an infinite-dimensional generalization thereof explained in [26], section 4.

The Poisson brackets among the  $a_m$  following from (5.21) are essentially those of harmonic oscillators,

$$\{a_m, a_n\}_{PB} = \frac{i\delta_{m,-n}}{m(m^2 - s^2)} \quad \text{for } m, n, \neq 0, \pm s \quad (5.25)$$

If the Darboux-like coordinates (5.20) and the expression for the Virasoro charges (5.18) are known, we can in principle express the Virasoro charges in terms of the  $a_m$ , yielding a perturbation series in powers of  $1/c$ .

Now we turn to the issue of quantizing the conical coadjoint orbits. The  $s = 1$  case, which is the orbit of global AdS, can be quantized using standard methods [26] (it can be given a Kähler structure) and gives the unitary representation based on the  $SL(2, \mathbb{R})$  invariant vacuum. Hence we recover the standard result that global AdS is dual to the  $SL(2, \mathbb{R})$  invariant vacuum in the CFT.

On the other hand, the  $s > 1$  orbits, corresponding to the conical excesses, have so far defied quantization using standard methods [26]. This is most likely related to their being only saddle points of the energy: we don't expect there to be any way of quantizing such solutions in a way that leads to unitary highest weight Virasoro representations. Nevertheless, we will argue that they do give rise to a nonunitary highest weight Virasoro representation acting on a Fock space, which we will construct as a perturbation expansion in  $1/c$ . This, then, is what we will mean by 'quantization' of these orbits.

We start by replacing the classical coordinates  $a_m$  on the orbit by quantum operators  $A_m$  satisfying commutation relations obtained from (5.25) by sending  $-i\{\cdot, \cdot\}_{PB} \rightarrow [\cdot, \cdot]$ :

$$[A_m, A_n] = \frac{\delta_{m,-n}}{m(m^2 - s^2)}. \quad (5.26)$$

From the classical expressions (5.18) we can expect that the quantum Virasoro generators, which we will denote by  $L_m$ , can be represented as composite operators in terms of the oscillators  $A_m$ , with some as yet unknown ordering prescription. Let's start with the Virasoro energy  $L_0$ , which to our current level of accuracy can be written as

$$L_0 = -\frac{cs^2}{24} + n_0 + \sum_{m \in \mathbb{N}} m^2(m^2 - s^2) A_{-m} A_m + \mathcal{O}(c^{-1/2}) \quad (5.27)$$

where we have introduced an ordering constant  $n_0$  of order  $c^0$  reflecting the ordering ambiguity in the the quadratic term. From its commutator with  $A_{\pm m}$

$$[L_0, A_{\pm m}] = \mp m A_{\pm m} + \mathcal{O}(c^{-1/2}) \quad (5.28)$$

we see that, for large  $c$ , the  $A_m$  with  $m > 0$  lower the  $L_0$  eigenvalue, while the  $A_m$  with  $m < 0$  act as raising operators. Note that the oscillators with modes  $|m| < s$  have a nonstandard minus sign in both the energy (5.27) and the commutation relations (5.26) as a consequence of expanding around a saddle point. These modes are similar to those arising in the matter CFT of string theory from the timelike field  $X^0$  with negative kinetic energy. As is the case for that theory, we need to make some choice for the 'vacuum' state by stating which of the  $A_m$  with  $|m| < s$  annihilate it. No choice is free from unpleasant

features: we must either give up having an  $L_0$  which is bounded below or having only positive norm states. We will here make the choice to keep the energy levels positive by taking our the Fock vacuum  $|0\rangle_s$  to be the state annihilated by the lowering operators,

$$A_m|0\rangle_s = 0 \quad \text{for } m > 0. \quad (5.29)$$

By acting on this vacuum with the raising operators, we build up a Fock space in the usual way. We require the inner product on this Fock space to satisfy

$$A_m^\dagger = A_{-m} \quad (5.30)$$

reflecting the reality condition (5.22) and assume the ground state to be normalized as  ${}_s\langle 0|0\rangle_s = 1$ . For  $s > 1$ , our Fock space contains negative norm states, since we must have  $\|A_{-1}|0\rangle_s\|^2 = 1/(1-s^2) < 0$ . The problems with normalizability are also evident from the expression for the position space wavefunction

$$\Psi_s^0(x) \equiv \langle x|0\rangle_s \sim e^{-\frac{1}{2}\sum_{m \in \mathbb{N}_0} \text{sgn}(m-s)x_m^2} \quad (5.31)$$

i.e. the modes with  $m < s$  come with a wrong sign in the Gaussian wavefunction.

Let's discuss some properties of the expression for the Virasoro generators  $L_m$  in terms of the oscillators  $A_n$ . We will choose to represent the  $L_m$  as creation-annihilation normal ordered expressions in the oscillators satisfying  $L_m^\dagger = L_{-m}$ , introducing unknown normal ordering constants where necessary. These constants are then to be fixed by requiring that the  $L_m$  obey the quantum Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m^3\delta_{m,-n}. \quad (5.32)$$

We will comment on a systematic procedure to achieve this below. Since the  $L_m$  are normal ordered and of level  $m$  in the oscillators, it follows that we will automatically have

$$L_m|0\rangle_s = 0 \quad m > 0 \quad (5.33)$$

i.e. our Fock space will furnish a highest weight (i.e. primary) representation of the Virasoro algebra, which is however nonunitary because of the negative norm states mentioned above. In the following we will determine precisely which nonunitary representations the conical spaces correspond to.

To the order at which we have been working, only  $L_0$  requires the introduction of a normal ordering constant ( $n_0$  in (5.27)), and the remaining Virasoro generators are given by

$$L_m = -i\sqrt{\frac{c}{12}}m(m^2 - s^2)A_m + \mathcal{O}(c^0) \quad \text{for } m, \neq 0, \pm s \quad (5.34)$$

$$L_{\pm s} = \sum_{m \in \mathbb{N} + \nu_s} 2^{-\delta_{m,0}} \left(m^2 - \frac{s^2}{4}\right) \left(m^2 - \frac{9s^2}{4}\right) A_{-m \pm \frac{s}{2}} A_{m \pm \frac{s}{2}} + \mathcal{O}(c^{-1/2}) \quad (5.35)$$

where we have introduced a constant  $\nu_s$  which is 0 for  $s$  even and  $1/2$  for  $s$  odd:

$$\nu_s \equiv \frac{s \bmod 2}{2}. \quad (5.36)$$

Before turning to the determination of the normal ordering constant  $n_0$ , we outline our proposed strategy for quantizing the conical orbits in a semiclassical expansion in  $1/c$ . We have seen that Virasoro generators admit an expansion in powers of  $1/c$  in terms of the oscillators  $A_m$  (given by our formulas (5.27,5.34,5.35) plus corrections), where we consider the oscillators  $A_m$  to be of ‘order  $c^0$ ’ as the commutation relations (5.26) are  $c$ -independent. We introduce the operators  $\mathcal{F}_{m,n}$  measuring the failure of the Virasoro algebra to hold:

$$\mathcal{F}_{m,n} = [L_m, L_n] - (m-n)L_{m+n} - \frac{c}{12}m^3\delta_{m,-n}. \quad (5.37)$$

When expressed in terms of the oscillators  $A_m$  these admit an expansion in powers of  $1/c$ :

$$\mathcal{F}_{m,n} = \sum_{\alpha \in \mathbb{N}/2} \frac{\mathcal{F}_{m,n}^{(\alpha-2)}}{c^{\alpha-2}}. \quad (5.38)$$

with  $\mathcal{F}_{m,n}^{(\alpha)}$  independent of  $c$ . We expect that, by adjusting a finite number of normal ordering constants up to some order  $(1/c)^\beta$ , it will be possible to make the Virasoro commutation relations hold to order  $(1/c)^\beta$  in the sense that

$$\mathcal{F}_{m,n}^{(\alpha)} = 0 \quad \text{for } -2 \leq \alpha \leq \beta. \quad (5.39)$$

Proceeding in this way we build up an oscillator realization of the Virasoro algebra, order by order in  $1/c$ . Although we don’t have a general proof that this procedure is consistent, we will see that it gives sensible and satisfying results at the first nontrivial order.

It is instructive to check the vanishing of the first few  $\mathcal{F}_{m,n}^{(\alpha)}$  using (5.27,5.34,5.35) and the commutation relations (5.26). The operators  $\mathcal{F}_{m,n}^{(-2)}$  and  $\mathcal{F}_{m,n}^{(-3/2)}$  vanish trivially as they are given by commutators with the  $c$ -number term in  $L_0$ . For the operators  $\mathcal{F}_{m,n}^{(-1)}$ ,  $\mathcal{F}_{m,0}^{(-1/2)}$  and  $\mathcal{F}_{m,s}^{(-1/2)}$  one finds

$$\begin{aligned} \mathcal{F}_{m,n}^{(-1)} &= -\frac{1}{12}m(m^2-s^2)n(n^2-s^2)[A_m, A_n] - \frac{m}{12}(m^2-s^2)\delta_{m,-n} \\ \mathcal{F}_{m,0}^{(-1/2)} &= -\frac{i}{\sqrt{12}}m(m^2-s^2)\sum_{n \in \mathbb{N}} n^2(n^2-s^2)[A_m, A_{-n}A_n] + \frac{im^2(m^2-s^2)}{\sqrt{12}}A_m \\ \mathcal{F}_{m,s}^{(-1/2)} &= -\frac{i}{\sqrt{12}}m(m^2-s^2)\sum_{n \in \mathbb{N}+\nu_s} 2^{-\delta_{n,0}}\left(n^2-\frac{s^2}{4}\right)\left(n^2-\frac{9s^2}{4}\right)[A_m, A_{-n+\frac{s}{2}}A_{n+\frac{s}{2}}] \\ &\quad + \frac{i}{\sqrt{12}}(m^2-s^2)((m+s)^2-s^2)A_{m+s}. \end{aligned} \quad (5.40)$$

These operators do not involve any normal ordering constants and hence their vanishing is guaranteed because the classical Virasoro algebra holds by construction. It is straightforward to verify that they indeed vanish upon using (5.26) and the commutator

$$[A_m, A_{-n+\frac{s}{2}}A_{n+\frac{s}{2}}] = \frac{\delta_{m,n-\frac{s}{2}} + \delta_{m,-n-\frac{s}{2}}}{m(m^2-s^2)}A_{m+s} \quad (5.41)$$

which follows from (5.26).

## 5.4 One-loop energy correction

The first nontrivial check on our proposed method to quantize the conical orbits comes at order  $c^0$ , where the operator  $\mathcal{F}_{s,-s}^{(0)}$  involves the normal ordering constant  $n_0$  in (5.27). Since  $c$  multiplies the action (2.2),  $1/c$  is the loop-counting parameter in our theory and the constant  $n_0$ , which is of order  $c^0$ , can be interpreted as the 1-loop correction to the energy of the conical solutions. The quickest way to determine  $n_0$  is to impose that the expectation value of  $\mathcal{F}_{s,-s}^{(0)}$  in the Fock vacuum vanishes:

$${}_s\langle 0 | \mathcal{F}_{s,-s}^{(0)} | 0 \rangle_s = 0. \quad (5.42)$$

This leads to

$$n_0 = \frac{1}{2s} {}_s\langle 0 | [L'_s, L'_{-s}] | 0 \rangle_s. \quad (5.43)$$

where  $L'_s$  ( $L'_{-s}$ ) is the piece of  $L_s$  ( $L_{-s}$ ) in (5.35) which involves two annihilation (creation) operators:

$$L'_{\pm s} = \sum_{m=\nu_s}^{s/2-1} 2^{-\delta_{m,0}} \left( m^2 - \frac{s^2}{4} \right) \left( m^2 - \frac{9s^2}{4} \right) A_{-m \pm \frac{s}{2}} A_{m \pm \frac{s}{2}}. \quad (5.44)$$

From this expression we can evaluate (5.43) using (5.26) and obtain

$$n_0 = \sum_{m=\nu_s}^{s/2-1} 2^{-\delta_{m,0}} \left( \frac{9s}{8} - \frac{m^2}{2s} \right) \quad (5.45)$$

$$= \frac{1}{24} (s-1)(1+13s). \quad (5.46)$$

It remains to check that, with the value (5.46) of  $n_0$ , the operator  $\mathcal{F}_{s,-s}^{(0)}$  indeed vanishes. It is given by

$$\begin{aligned} \mathcal{F}_{s,-s}^{(0)} = & \sum_{m,n \in \mathbb{N} + \nu_s} \left( 2^{-\delta_{m,0} - \delta_{n,0}} \left( m^2 - \frac{s^2}{4} \right) \left( m^2 - \frac{9s^2}{4} \right) \left( n^2 - \frac{s^2}{4} \right) \left( n^2 - \frac{9s^2}{4} \right) \times \right. \\ & \left. [A_{-m+\frac{s}{2}} A_{m+\frac{s}{2}}, A_{-n-\frac{s}{2}} A_{n-\frac{s}{2}}] \right) - 2sn_0 - 2s \sum_{m \in \mathbb{N}} m^2 (m^2 - s^2) A_{-m} A_m \end{aligned} \quad (5.47)$$

and can be shown to vanish upon using

$$\begin{aligned} [A_{-m+\frac{s}{2}} A_{m+\frac{s}{2}}, A_{-n-\frac{s}{2}} A_{n-\frac{s}{2}}] = & \left( \frac{A_{-m+\frac{s}{2}} A_{m-\frac{s}{2}}}{(m+\frac{s}{2})((m+\frac{s}{2})^2 - s^2)} + \frac{A_{-m-\frac{s}{2}} A_{m+\frac{s}{2}}}{(-m+\frac{s}{2})((m-\frac{s}{2})^2 - s^2)} \right) \delta_{m,n} \\ & + \left( \frac{A_{m+\frac{s}{2}} A_{-m-\frac{s}{2}}}{(-m+\frac{s}{2})((m-\frac{s}{2})^2 - s^2)} + \frac{A_{m-\frac{s}{2}} A_{-m+\frac{s}{2}}}{(m+\frac{s}{2})((m+\frac{s}{2})^2 - s^2)} \right) \delta_{m,-n}. \end{aligned} \quad (5.48)$$

Similarly one verifies that  $\mathcal{F}_{s,0}^{(0)}$  is zero as well. To check the vanishing of the remaining order  $c^0$  operators  $\mathcal{F}_{m,n}^{(0)}$  would require going to the next order in our expansion (5.34) and would involve determining further integration constants, which we will leave for further study.

Summarizing, we have computed the Virasoro energy of the conical spaces to 1-loop order to be

$$\boxed{{}_s\langle 0|L_0|0\rangle_s = -\frac{s^2 c}{24} + \frac{1}{24}(s-1)(1+13s) + \mathcal{O}(c^{-1})}. \quad (5.49)$$

### 5.5 Null states and Kac's degenerate representations

We have seen in section 5.2 that the coadjoint orbits of the conical solutions are rather special in that they have a three parameter isotropy group  $SO(2,1)^{(s)}$ . As a consequence, the classical Virasoro charges  $l_0, l_s, l_{-s}$  on the coadjoint orbit can be expressed in terms of the other  $l_m$ , see (5.18). It is natural to expect that also on the quantum level the corresponding Virasoro representation is such that  $L_0, L_s, L_{-s}$  can be expressed in terms of the remaining  $L_m$  [26]. This property is the hallmark of a Virasoro representation which contains a null vector at level  $s$ : acting with the vanishing combination of  $L_{-s}$  and the other generators on the Fock vacuum we obtain a null vector, and conversely, in a highest weight representation with a null vector at level  $s$  it is possible to express  $L_0, L_{\pm s}$  as a function of the remaining  $L_m$  (see [26] for a proof).

For definiteness, from (5.35) we see that, to our level of accuracy, we can express  $L_0, L_s, L_{-s}$  as

$$L_0 = -\frac{cs^2}{24} + n_0 + \frac{12}{c} \sum_{m \in \mathbb{N} \setminus \{0, s\}} \frac{L_{-m} L_m}{m^2 - s^2} + \mathcal{O}(c^{-3/2}) \quad (5.50)$$

$$L_{\pm s} = \frac{12}{c} \sum_{\substack{m \in \mathbb{N} + \nu_s \\ m \neq s/2, 3s/2}} \frac{2^{-\delta_{m,0}} L_{-m \pm s/2} L_{m \pm s/2}}{m^2 - \frac{s^2}{4}} + \mathcal{O}(c^{-3/2}) \quad (5.51)$$

The null vector obtained by acting with (5.51) on the Fock vacuum is, after a shift in the summation variable and a reordering of terms:

$$\boxed{\left( L_{-s} - \frac{6}{c} \sum_{m=1}^{s-1} \frac{L_{-m} L_{m-s}}{m(m-s)} + \mathcal{O}(c^{-3/2}) \right) |0\rangle_s = 0} \quad (5.52)$$

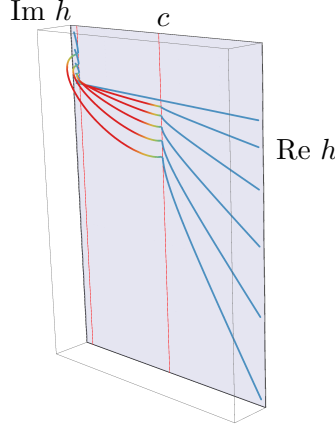
This expression is the operator version of a classical formula derived in [26]. The second term is again (minus) the operator  $L'_{-s}$  defined in (5.44).

Virasoro representations containing a null vector are referred to as degenerate and were classified by Kac [32] (see also [39]). We will now identify precisely which Kac representations the quantized conical spaces correspond to. Kac's result can be summarized as follows: for any value of the central charge  $c$ , there is a degenerate representation for every pair of nonzero natural numbers<sup>16</sup>  $(r, s)$ . It contains a null vector at level  $rs$  and is based on a primary of weight

$$h_{(r,s)} = -\frac{1}{24} - \frac{rs}{2} + \frac{1}{48}(13-c)(r^2 + s^2) + \frac{1}{48}\sqrt{(1-c)(25-c)}(r^2 - s^2). \quad (5.53)$$

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<sup>16</sup>For small values of the central charge some of those representations can become equivalent, but this doesn't play a role in the large  $c$  regime we are considering.



**Figure 1:** The conformal weights  $h_{(1,s)}$  for  $1 \leq s \leq 6$  as a function of the central charge  $c$ . The red lines correspond to  $c = 1$  and  $c = 25$ , between which the conformal weights become complex.

The degenerate representations are usually only considered at small values of the central charge,  $0 < c < 1$ , since only in that regime they have a conformal weights above the vacuum and stand a chance of being unitary. However, they exist as nonunitary representations also for  $c$  outside this range. Note that, for  $r \neq s$ ,  $h_{(r,s)}$  becomes complex in the range  $1 < c < 25$  and is real again for  $c \geq 25$ . Comparing the large  $c$  expansion

$$h_{(r,s)} \sim -\frac{s^2 c}{24} + \frac{1}{24} (-12rs + 13s^2 - 1) + \mathcal{O}(c^{-1}) \quad (5.54)$$

with (5.49) leads to the unambiguous identification of the quantized orbit of the conical space labelled by  $s$  with the degenerate representation of type  $(1, s)$  of the left-moving Virasoro algebra. The conformal weight of the representations  $h_{(1,s)}$  is plotted as a function of  $c$  in figure 1. Similarly, the right-moving winding number  $\tilde{s}$  corresponds to the representation of type  $(1, \tilde{s})$  of the right-moving Virasoro algebra.

A further check comes from comparing the null vectors: the explicit form of the null vector at level  $s$  of the  $(1, s)$  degenerate representation was first derived by Benoit and Saint-Aubin [40]. Their expression (see (2.7) in [40]) is written as a sum over partitions of  $s$ , and the leading contributions at large  $c$  come from the partitions with one and two elements. It is straightforward to check that these contributions coincide with our expression (5.52). This identification between the conical states and  $(1, s)$  degenerate representations was already proposed in [20] based on classical considerations in the bulk, and we have hereby confirmed agreement also on the 1-loop level.

## 6. Discussion

We end with some comments and directions for future work.

- We argued that the conical spaces play a role in nonunitary versions of holography and are dual to primaries of degenerate Virasoro representations. The origin of the nonunitary behaviour came from expanding around a saddle point of the Virasoro

energy. This seems to be the generic situation in the known consistent nonunitary quantum field theories. For example in Vafa’s nonunitary holographic theories [18], which are based on large  $N$  gauge theories where the gauge group is a supergroup, the classical ground state in the super matrix model description also has negative energy directions. We also want to remark that our Fock vacua  $|0\rangle_s$  seem rather similar to the Kodama state in 4D quantum gravity [41], which is a saddle point of the Hamiltonian as was pointed out in [42]. Our results suggest that this state may yet play a role in examples of nonunitary holography.

- We computed the 1-loop energy correction for conical spaces from the bulk perspective using operator methods, by quantizing the natural Poisson bracket on coadjoint orbits. It would be instructive to rederive this result from a 1-loop determinant in the path integral formalism. Here the challenge is to derive the action governing the boundary graviton fluctuations. A natural guess for this action is the so-called geometric action [27] which reproduces the Poisson bracket on the coadjoint orbits by construction, and which should be obtainable from the original Chern-Simons action (some evidence for this appears in [27],[28]).
- The method outlined in section 5 allows one to derive a perturbative expression for the null vector in the degenerate representation in a large  $c$ -expansion. The expression for the exact null vectors [40] can similarly be derived from a large  $c$  expansion [43], but the relation between the two approaches is as yet unclear. Clarifying this link would be especially interesting since the knowledge of the exact null vector implies knowledge of the energy correction to all orders.
- We provided evidence that the Chern-Simons formulation of 3D gravity accommodates the degenerate Virasoro representations of the type  $(1, s)$ . While a CFT containing only these primaries does have a spectrum which is closed under OPE’s, our experience with minimal model CFTs at  $c < 1$  suggests that such a theory cannot be modular invariant and that this requires the inclusion of the more general  $(r, s)$  representations. It therefore natural to ask what type of matter has to be added to pure gravity in order to get all  $(r, s)$  representations in the spectrum. There is evidence [20] that the theory that accomplishes this is the Prokushkin-Vasiliev theory [44] at the value  $\lambda = 2$  of the vacuum parameter, where it reduces to gravity coupled to a somewhat unusual scalar field, whose excitations around the conical spaces give the  $(r, s)$  representations. In fact, our computation of the energy shift (5.49) was the missing ingredient in the matching of the bulk 1-loop partition function to the partition function of such a CFT at large  $c$ . It is an interesting open question whether the full CFT partition function including  $1/c$  corrections is (presumably in some formal sense) modular invariant, and whether it can be reproduced from the bulk side.
- The degenerate Virasoro representations display various interesting features at small values of the central charge ( $0 < c < 1$ ), which would be very interesting to un-



derstand physically as strong coupling phenomena in the bulk<sup>17</sup>. Examples are the fact that some of the  $(r, s)$  representations can become equivalent to each other and become unitary at special values of the central charge in the strong coupling regime  $0 < c < 1$ . For those special values of  $c$  a truncation of the spectrum of  $(r, s)$  representations gives a unitary modular invariant theory, leading to the Virasoro minimal models, some of which may be interpretable as strongly coupled gravity theories [45].

- As already mentioned in the Introduction, the conical solutions in pure gravity which we studied in this paper are the more tractable cousins of similar solutions<sup>18</sup> [19] in higher spin theories with asymptotic  $W$ -symmetry [46],[47]. It would be interesting to extend the regularity arguments of section 4 using point particle probes to the higher spin case, as well to understand whether they, too, are singular in the metric-like formulation of the theory [48]. For the higher spin conical solutions, a similar holographic interpretation as degenerate nonunitary representations of the  $W$ -algebra was proposed in [20]. The arguments for this identifications were purely classical, and it would be interesting to generalize our method to compute quantum corrections to these solutions. While the concepts of finite  $W$ -symmetry transformations and coadjoint orbits are certainly much less understood than their Virasoro counterparts, we do feel that it should be possible to extend the perturbative setup of the current work the higher spin case.

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<sup>17</sup>As we reviewed in section 2, with our choice of gauge group  $c$  is quantized in units of 24, so to consider small values of  $c$  we should either consider a covering group or work in Euclidean signature, where  $c$  is not quantized.

<sup>18</sup>The interpretation as topological solitons characterized by a winding number seems to be special to the pure gravity case, since the higher spin theories are based on the gauge groups  $SL(N, \mathbb{R})$  (for the Lorentzian theory) or  $SL(N, \mathbb{C})$  (for the Euclidean theory), which have  $\pi_1(SL(N, \mathbb{R})) = \mathbb{Z}_2$  and  $\pi_1(SL(N, \mathbb{C})) = 1$ .

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